

Exercise 100

If $y = f(u)$ and $u = g(x)$, where f and g possess third derivatives, find a formula for d^3y/dx^3 similar to the one given in Exercise 99.

Solution

Use the chain rule to differentiate y .

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(y) \\ &= \frac{d}{dx}[f(u)] \\ &= f'(u) \cdot \frac{d}{dx}(u) \\ &= \frac{df}{du} \frac{du}{dx}\end{aligned}$$

Find the second derivative now.

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dx} \right) \\ &= \frac{d}{dx} \left(\frac{df}{du} \frac{du}{dx} \right) \\ &= \left[\frac{d}{dx} \left(\frac{df}{du} \right) \right] \frac{du}{dx} + \frac{df}{du} \left[\frac{d}{dx} \left(\frac{du}{dx} \right) \right] \\ &= \left[\frac{du}{dx} \frac{d}{du} \left(\frac{df}{du} \right) \right] \frac{du}{dx} + \frac{df}{du} \left[\frac{d}{dx} \left(\frac{du}{dx} \right) \right] \\ &= \left[\frac{d}{du} \left(\frac{df}{du} \right) \right] \left(\frac{du}{dx} \right)^2 + \frac{df}{du} \left[\frac{d}{dx} \left(\frac{du}{dx} \right) \right] \\ &= \frac{d^2f}{du^2} \left(\frac{du}{dx} \right)^2 + \frac{df}{du} \frac{d^2u}{dx^2} \\ &= \frac{d^2y}{du^2} \left(\frac{du}{dx} \right)^2 + \frac{dy}{du} \frac{d^2u}{dx^2}\end{aligned}$$

Find the third derivative now.

$$\begin{aligned}
 \frac{d^3y}{dx^3} &= \frac{d}{dx} \left(\frac{dy^2}{dx^2} \right) \\
 &= \frac{d}{dx} \left[\frac{d^2y}{du^2} \left(\frac{du}{dx} \right)^2 + \frac{dy}{du} \frac{d^2u}{dx^2} \right] \\
 &= \frac{d}{dx} \left[\frac{d^2y}{du^2} \left(\frac{du}{dx} \right)^2 \right] + \frac{d}{dx} \left(\frac{dy}{du} \frac{d^2u}{dx^2} \right) \\
 &= \left[\frac{d}{dx} \left(\frac{d^2y}{du^2} \right) \right] \left(\frac{du}{dx} \right)^2 + \frac{d^2y}{du^2} \left[\frac{d}{dx} \left(\frac{du}{dx} \right)^2 \right] \\
 &\quad + \left[\frac{d}{dx} \left(\frac{dy}{du} \right) \right] \frac{d^2u}{dx^2} + \frac{dy}{du} \left[\frac{d}{dx} \left(\frac{d^2u}{dx^2} \right) \right] \\
 &= \left[\frac{du}{dx} \frac{d}{du} \left(\frac{d^2y}{du^2} \right) \right] \left(\frac{du}{dx} \right)^2 + \frac{d^2y}{du^2} \left[2 \left(\frac{du}{dx} \right) \cdot \frac{d}{dx} \left(\frac{du}{dx} \right) \right] \\
 &\quad + \left[\frac{du}{dx} \frac{d}{du} \left(\frac{dy}{du} \right) \right] \frac{d^2u}{dx^2} + \frac{dy}{du} \left[\frac{d}{dx} \left(\frac{d^2u}{dx^2} \right) \right] \\
 &= \left[\frac{d}{du} \left(\frac{d^2y}{du^2} \right) \right] \left(\frac{du}{dx} \right)^3 + 2 \frac{d^2y}{du^2} \left[\left(\frac{du}{dx} \right) \cdot \left(\frac{d^2u}{dx^2} \right) \right] \\
 &\quad + \left[\frac{du}{dx} \left(\frac{d^2y}{du^2} \right) \right] \frac{d^2u}{dx^2} + \frac{dy}{du} \left[\left(\frac{d^3u}{dx^3} \right) \right] \\
 &= \frac{d^3y}{du^3} \left(\frac{du}{dx} \right)^3 + 2 \frac{d^2y}{du^2} \frac{du}{dx} \frac{d^2u}{dx^2} \\
 &\quad + \frac{d^2y}{du^2} \frac{du}{dx} \frac{d^2u}{dx^2} + \frac{dy}{du} \frac{d^3u}{dx^3}
 \end{aligned}$$